Minimax games, spin glasses, and the polynomial-time hierarchy of complexity classes

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(Received 17 December 1997)

We use the negative replica method, which was originally developed for the study of overfrustation in disordered systems, to investigate the statistical behavior of the cost function of minimax games. These games are treated as hierarchical statistical mechanical systems, in which one of the components is at negative temperature. [S1063-651X(98)09706-2]

PACS number(s): 64.60.Cn, 61.43.Fs, 05.20.−y, 05.50.+q

I. INTRODUCTION

The theory of spin glasses has found interesting applications in several branches of science [1]. In the theory of combinatorial optimization it inspired the invention of the so-called simulated annealing heuristic search technique [2]. With the help of the replica method, several authors [3–5] managed to obtain analytical insight into optimal solutions of some hard optimization problems. The interest in these studies was driven by the fact that many of these problems were members of the NP complexity class, which means that to check their solutions requires only polynomial time, but to find them is presumably much harder.

NP is among the first few members of a hierarchy of complexity classes of increasing difficulty, the polynomial-time hierarchy PH of Meyer and Stockmeyer [6]. Examples of problems from this hierarchy are adversary games, where the first player tries to minimize the objective function while the second one tries to maximize it. In this paper we treat the case in which one of the players has control over the spins of a spin glass, while the other controls the external magnetic field, and the objective function is the energy of the spin configuration.

The standard machinery of statistical mechanics provides information on the ground state (minimum of energy), as the temperature approaches zero. To study the maximum, we need to approach zero from negative direction. Fortunately, this step can be incorporated into the replica method by allowing the number of replicas to be negative. The method of negative replicas was invented by Dotsenko, Franz, and Mezard to study partial annealing and overfrustation in disordered systems [7]. (Some related works are [8–11].) We use this framework for the investigation of minimax games.

In Sec. II we give a short, nontechnical description of the polynomial-time hierarchy of complexity classes. In Sec. III we apply this extension of the replica method for three simple models. Section IV contains an extension of the negative replica method for multimove games.

II. THE POLYNOMIAL-TIME HIERARCHY

In this section we closely follow the exposition of Stockmeyer [6]. To formulate a rough definition of the complexity classes it is easier to use decision problems than optimization ones. We define the complexity class P as those problems which are solvable by a deterministic (and sequential) computer in time bounded by some polynomial of the size of the problem. Of course, one should spell out in a little more detail the kind of computers used (usually a Turing or Random Access Machine), however, the class P is remarkably stable with respect to changes of the computational model.

The definition of the class NP is similar, but in this case the use of nondeterministic computers is allowed. The nondeterministic model of computation is more powerful than the deterministic one. Let us take, for example, the most representative problem of NP, the satisfiability of an arbitrary Boolean expression. If there is an assignment of truth values to the variables of the expression such that the expression evaluates to ''true,'' then a nondeterministic computer is able to verify that in polynomial time. In the first few steps it correctly guesses that assignment, and then by a deterministic algorithm it verifies that the assignment indeed satisfies the Boolean expression. These steps take only polynomial time. So NP contains those problems, whose solutions, if they exist, can be checked in polynomial time. A basic conjecture of computer science is that the inclusion $P\subset NP$ is proper, i.e., there are problems easy to check but hard to solve.

In the case of spin glasses the decision problem is that, given a J_{ij} coupling constants matrix and a number K, is there any spin configuration s_i such that $E_J(s_i) =\Sigma_{i,j}J_{ij}s_is_j\leq K$? [To keep the size of the problem under control, J_{ij} should take only discrete (maybe ± 1) values.] A closely related problem class is co-NP, the complement of NP. Here the task is to recognize those problems which have no solution. For example, in the spin-glass case one needs to prove that there is no spin configuration with energy less than a given constant. It is unlikely that such proof of polynomial length exists for a random J_{ij} matrix , so it is believed that $NP\neq$ co-NP. Optimization problems require the ability to solve both NP and co-NP problems. To prove that U_0 is the minima of $E_J(s)$, one should first find s such that $U_0=E_J(s)$, then solve a co-NP problem proving that there is no such s that $E_J(s)<U_0$.

Several ways exist to obtain problems harder than NP. The most obvious is to allow more (say exponential) time for the computation. A more subtle way to increase the power of the computational model is the use of oracle machines. They have an additional instruction ''Call-Oracle.'' When the machine executes this instruction, it presents the oracle a problem from the oracle's problem class for which the oracle returns the solution or gives a ''no'' answer in a single step. The power of an oracle computer depends on the oracle's problem class C. Since the oracle recognizes nonmembership in C, too, the oracles C and co-C have the same computational power. In this manner, $NP(C)$ $[P(C)]$ is defined as the decision problem class, whose satisfiability can be decided by a nondeterministic (deterministic) computer with oracle C in polynomial time.

By denoting $P=\Sigma_0^P$, the polynomial-time hierarchy is defined as

$$\sum_k^P = NP\left(\sum_{k-1}^P\right), \quad \Delta_k^P = P\left(\sum_{k-1}^P\right), \quad \prod_k^P = \text{co}-\sum_k^P . \tag{1}$$

Members of this hierarchy occur in problems involving the alternation of existential and universal quantifiers. The satisfiability of the Boolean formula $f(\mathbf{x})$ (i.e., $f\in NP$) means $\exists \mathbf{x} f(\mathbf{x})$, while its nonsatisfiability (i.e., $f\in \text{co}-NP$) is the same as $\forall \mathbf{x} \neg f(\mathbf{x})$. Boolean formulas $\exists \mathbf{x}_1 \forall \mathbf{x}_2 \cdots \exists \mathbf{x}_{2l+1} f(\mathbf{x}_1,\mathbf{x}_2,\dots)$ or $\exists \mathbf{x}_1 \forall \mathbf{x}_2 \cdots \forall \mathbf{x}_{2l} \neg f(\mathbf{x}_1,\mathbf{x}_2,\dots)$ with $(k-1)$-fold alternation of existential and universal quantifiers provides natural examples for problems from Σ_k^P. The determination of the satisfiability of such formulas can be described as a game between two adversary players. The first player's objective is to satisfy the formula, while the second one tries to set the variables $\mathbf{x}_2$, $\mathbf{x}_4$,.., so that the formula is not satisfied.

An optimization problem from the polynomial-time hierarchy is the determination of the outcome

$$M = \max_{\mathbf{x}_1} \min_{\mathbf{x}_2} \cdots c(\mathbf{x}_1,\mathbf{x}_2,\dots) \tag{2}$$

of a minimax game. For many functions $c(\mathbf{x}_1,\mathbf{x}_2,\dots)$, the computation of M is a Δ_{k+1}^P type problem if there are $k-1$ alternations of the min and max operators. In the next section we treat the case where $\mathbf{x}_1$ and $\mathbf{x}_2$ represent sets of discrete spin variables and $c(\mathbf{x}_1,\mathbf{x}_2)$ is the energy function of spin configurations.

III. SPIN GAMES

In this section we study two-move minimax games. The objective function is denoted by $H(u,v)$, where u and v are two sets of variables. The first player (the minimizer) controls the u variables, while the second one (the maximizer) controls the v variables. If both players play optimally, then the outcome of the game is

$$M = \inf_u [\sup_v H(u,v)]. \tag{3}$$

To apply the methods of statistical mechanics, $\inf_u h(u)$ $[\sup_v H(u,v)]$ is replaced by the free energy of a system with Hamiltonian h (H) at low positive (negative) temperature. For that purpose we introduce

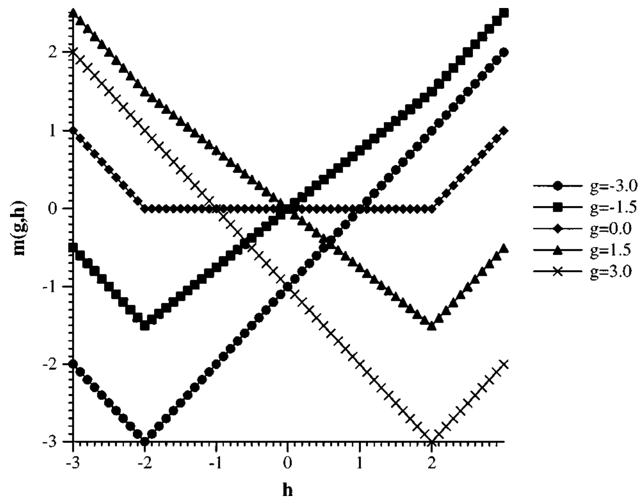

FIG. 1. Expected outcome of the game corresponding to the Hamiltonian (5) at optimal play.

$$\begin{aligned} M(\beta_u,\beta_v) &= -\frac{1}{\beta_u}\ln\sum_{\{u\}} \exp -\beta_u\left(\frac{1}{\beta_v}\ln\sum_{\{v\}} \exp\beta_v H(u,v)\right) \\ &= -\frac{1}{\beta_u}\ln\sum_{\{u\}} \left(\sum_{\{v\}} \exp\beta_v H(u,v)\right)^{-\beta_u/\beta_v} \\ &= -\frac{1}{\beta_u} \\ &\quad \times \lim_{n\to 0}\frac{1}{n}\left[\left(\sum_{\{u^a,v^\alpha\}} \exp\beta_v \sum_{\{a,\alpha\}} H(u^a,v^\alpha)\right)-1\right]. \end{aligned} \tag{4}$$

There are n replicas of u and $nk=-n\beta_u/\beta_v$ copies of v. If $\beta_u,\beta_v\to\infty$, then $M(\beta_u,\beta_v)\to M$ (at least if the zero temperature entropy vanishes, which is true even in the mean field theory of spin glasses).

To gain some experience with the method of negative replicas, we apply it first for the nonrandom Hamiltonian

$$H(u,v) = \frac{2}{N}\left(\sum_i u_i\right)\left(\sum_i v_i\right) + g\sum_i u_i + h\sum_i v_i, \quad i=1,...,N. \tag{5}$$

We are interested in the limit of $m(g,h) = \lim_{N\to\infty}\inf_u \sup_v H(u,v)/N$ (see Fig. 1). Since the Hamiltonian (5) is very simple, finding the optimal moves does not require too many computational resources. The player controlling the u variables only needs to find the minimum of Eq. (8), which can be done in constant time independently of N. Consequently this example (just as the next one) belongs to the complexity class P.

In this case the application of the $n\to 0$ limit is not necessary, so the u spins are not replicated. The partition function is

$$Z=\sum_{u,v^\alpha}\exp\beta_v\sum_\alpha H(u,v^\alpha)$$
$$=\int\frac{N\beta_v dx dy}{4i\pi}$$
$$\times\exp\left\{-N\left(\frac{\beta_v}{2}xy-\ln\{2\cosh[\beta_v k(g+x/k)]\}\right.\right.$$
$$\left.\left.-k\ln\{2\cosh[\beta_v(h+y)]\}\right)\right\}. \tag{6}$$

The large β saddle-point equations are

$$\frac{y_0}{2}=\tanh\left[-\beta_u\left(g+\frac{x_0}{k}\right)\right]\approx\text{sgn}\left(-g+\frac{x_0}{k}\right),$$
$$\frac{x_0}{2k}=\tanh[\beta_v(h+y_0)]\approx\text{sgn}(h+y_0). \tag{7}$$

Using $\ln(2\cosh\beta z)\approx\beta|z|$ for $\beta\gg 1$, one can check that

$$\lim_{\beta_u,\beta_v\to\infty}\frac{-1}{N\beta_u}\ln Z$$
$$=\min_{u\in[-1,1]}[2u\,\text{sgn}(2u+h)+gu+h\,\text{sgn}(2u+h)], \tag{8}$$

where the last expression is the outcome of the game if both players play optimally, since at optimal play $v=\text{sgn}(2u+h)$.

In the next example a random magnetic field has been added to the model:

$$H(u,v)=\frac{2}{N}\left(\sum_i u_i\right)\left(\sum_i v_i\right)+\sum_i (g+g_i)u_i$$
$$+\sum_i (h+h_i)v_i, \tag{9}$$

where $\overline{h_i}=\overline{g_i}=0$ and $\overline{h_i^2}=\overline{g_i^2}=1$. After some tedious but standard calculations, we obtain that $M(\beta_u,\beta_v)$ is equal to the saddle-point value of

$$-\frac{1}{\beta_u}\left(\frac{\beta_u}{2}pq+\int\frac{dz}{\sqrt{2\pi}}e^{-(1/2)z^2}\right.$$
$$\times\ln\{2\cosh[-\beta_u(\sqrt{J}p+g+z)]\}, \tag{10}$$
$$\left.-\frac{\beta_u}{\beta_v}\int\frac{dw}{\sqrt{2\pi}}e^{-(1/2)w^2}\ln\{2\cosh[\beta_v(\sqrt{J}q+h+w)]\}\right),$$

with respect to p and q. The saddle-point equations are

$$p_0=2\sqrt{J}\int\frac{dw}{\sqrt{2\pi}}e^{-(1/2)w^2}\tanh[\beta_v(\sqrt{J}q_0+h+w)],$$
$$q_0=2\sqrt{J}\int\frac{dz}{\sqrt{2\pi}}e^{-(1/2)z^2}\tanh[-\beta_u(\sqrt{J}p_0+g+z)]. \tag{11, 12}$$

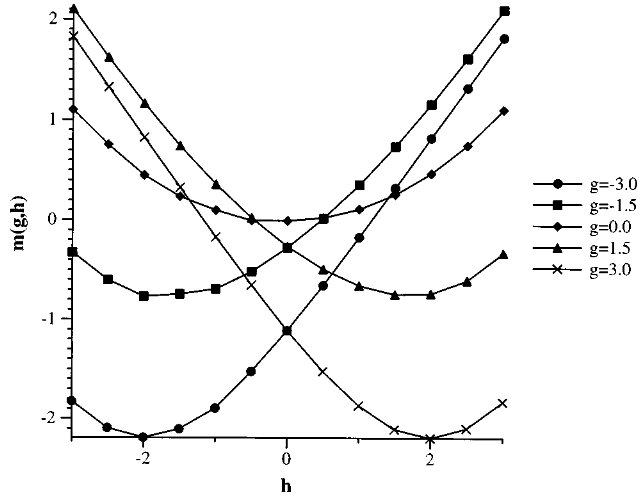

FIG. 2. . Expected outcome of the game corresponding to the Hamiltonian (9) at optimal play.

The numerical solution of these equations is presented in Fig. 2.

Since the Hamiltonian (9) is fairly simple, the expression (10) can be derived without the use of replicas, too. For that purpose, we assume that $M(\beta_u,\beta_v)$ receives its dominant contribution from spin configurations where the u spins' average magnetization is $\overline{u}$. Then

$$M(\beta_u,\beta_v)=\frac{-1}{\beta_u N}\int\prod_i\frac{dg_i}{\sqrt{2\pi}}e^{-(1/2\Sigma_i g_i^2)}$$
$$\times\ln\left\{\int d\lambda\sum_{\{u_i\}}e^{-i\lambda(\Sigma_i u_i-N\overline{u})}\right.$$
$$\left.\times\exp\left(-\beta_u\left[\sum_i (g+g_i)u_i+Nf_v(\overline{u})\right]\right)\right\}$$

where $f_v(\overline{u})$ is the free energy of the v_i spins in the external field of the u_i spin variables:

$$f_v(\overline{u})=\frac{1}{\beta_v}\int\frac{dw}{\sqrt{2\pi}}e^{-(1/2)w^2}\ln\{2\cosh[\beta_v(2J\overline{u}+h+w)]\}. \tag{13}$$

$M(\beta_u,\beta_v)$ evaluates to

$$\frac{-1}{\beta_u}\left[i\lambda\overline{u}+\int\frac{dz}{\sqrt{2\pi}}e^{-(1/2)z^2}\ln\{2\cosh[-\beta_u(g+z)+i\lambda]\}\right.$$
$$\left.-\frac{\beta_u}{\beta_v}\int\frac{dw}{\sqrt{2\pi}}e^{-(1/2)w^2}\ln\{2\cosh[\beta_v(2J\overline{u}+h+w)]\}\right]. \tag{14}$$

This formula should be computed at its saddle-point value with respect to λ and its minimum with respect to $\overline{u}\in[-1,1]$. After the change of variables $\overline{u}=q/(2\sqrt{J})$ and $i\lambda=\beta_u\sqrt{J}p$, expressions (10) and (14) coincide. Since we managed to evaluate $M(\beta_u,\beta_v)$ without the use of replicas, too, this example is certainly not the most impressive application of the negative replica method. Nevertheless, this model provides an example where one can analytically prove that the replica method works.

Finally, we attempt to treat the case of a spin-glass type objective function

$$H_J(s_i,h_i)=\sum_{1\leqslant i\leqslant j\leqslant N} J_{ij}s_is_j +g\sum_{1\leqslant i\leqslant N} h_is_i, \quad h_i=\pm 1, \quad s_i=\pm 1 \quad (15)$$

where J_{ij} is a random variable with Gaussian distribution

$$d\mu(J_{ij})=\sqrt{N/2J}\exp(-J^2/2N)dJ_{ij}. \quad (16)$$

This problem seems to belong to the complexity class Δ_2^P, since the maximizer needs to solve the problem of the maximization of the spin-glass energy function (15), and this problem is $NP\cap$CO-NP.

The minimizer makes the first move and controls the h_i variables, while the maximizer makes the second move and controls the s_i spins. The partition function of this system is

$$\begin{aligned}Z_{n,k}&=\int\prod_{i<k} d\mu(J_{ik})\sum_{\{h_i^a,s_i^{a\alpha}\}}\exp\beta_v\sum_{a\alpha} H_J(h_i^a,s_i^{a\alpha})\\&=\int\prod_{a\alpha<b\beta}\left(\sqrt{\frac{N\beta}{2\pi}}dQ_{a\alpha b\beta}\right)\exp-N\Bigg\{-nk\frac{\beta_v^2}{4}+\frac{\beta_v^2}{2}\sum_{a\alpha<b\beta}Q_{a\alpha b\beta}^2\\&\quad-\ln\sum_{\{S^{a\alpha},H^a\}}\exp\beta_v\left[\beta_v\sum_{a\alpha<b\beta}Q_{a\alpha b\beta}S^{a\alpha}S^{b\beta}-g\sum_a H^a\sum_\alpha S^{a\alpha}\right]\Bigg\},\end{aligned}$$

where $k=-\beta_u/\beta_v$. In the one stage replica symmetry breaking approximation

$$Q_{a\alpha a\beta}=p, \quad \text{while} \quad Q_{a\alpha b\beta}=q \quad \text{for } a\neq b, \quad (17)$$

$$\begin{aligned}Z_{n,k}&=\int\prod_{a\alpha<b\beta}\left(\sqrt{\frac{N\beta}{2\pi}}dQ_{a\alpha b\beta}\right)\exp-N\Bigg\{\beta_v^2\left(-\frac{nk}{4}+\frac{n(n-1)k^2}{4}q^2+\frac{nk(k-1)}{4}p^2+\frac{nk}{2}p\right)-\ln\int\frac{dx}{\sqrt{2\pi}}e^{-(1/2)x^2}\\&\quad\times\left(\int\frac{dy}{\sqrt{2\pi}}e^{-(1/2)y^2}(\{2\cosh[\beta_v(\sqrt{q}x+\sqrt{p-q}y+g)]\}^k+\{2\cosh[\beta_v(\sqrt{q}x+\sqrt{p-q}y-g)]\}^k)\right)^n\Bigg\}.\end{aligned}$$

From this equation one obtains the expected outcome of the game:

$$\begin{aligned}m&=\frac{1}{N}\min_{\{h_i\}}[\max_{\{s_i\}}H_J(s_i,h_i)]=\lim_{n\to 0,\beta_v\to\infty}\frac{1}{k\beta_v nN}(Z_{n,k}-1)\\&=\frac{\beta_v}{4}[1+kq^2+(1-k)p^2-2p]+\frac{1}{k\beta_v}\int\frac{dx}{\sqrt{2\pi}}e^{-(1/2)x^2}\\&\quad\times\ln\int\frac{dy}{\sqrt{2\pi}}e^{-(1/2)y^2}(\{2\cosh[\beta_v(\sqrt{q}x+\sqrt{p-q}y+g)]\}^k+\{2\cosh[\beta_v(\sqrt{q}x+\sqrt{p-q}y-g)]\}^k),\end{aligned}$$

where the last expression should be evaluated at its saddle point. This expression is very similar to the free energy of a spin glass at the one stage replica symmetry breaking approximation [12]. Indeed, $Q_{a\alpha b\beta}$ might be regarded as an $nk\times nk$ matrix broken into blocks of size $k\times k$. However, here k is a fixed negative number. $m(p,q)$ has a minimum at $p=q=1$ on the line $p=q$. In this approximation

$$m_{\text{minimax}}(g)=\int\frac{dx}{\sqrt{2\pi}}e^{-(1/2)x^2}\min(|x+g|,|x-g|). \quad (18)$$

A better approximation is achieved if we search for the saddle point on the (p,q) plane. Since the first term of m

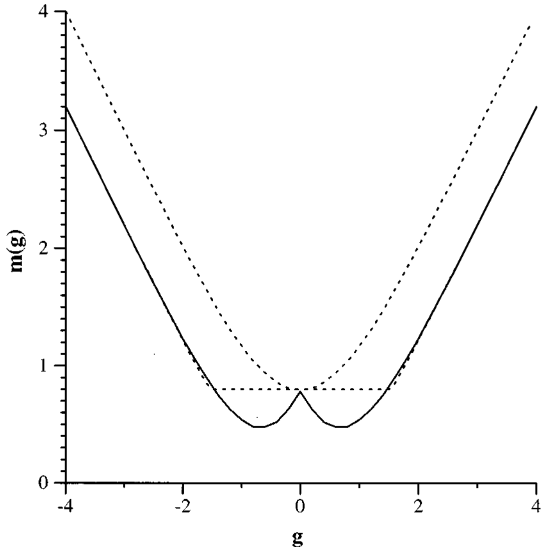

FIG. 3. Expected outcome of the game corresponding to the spin-glass type Hamiltonian (15) at optimal play. The dotted lines are exact upper and lower bounds, while the solid line is the result obtained in the one stage replica symmetry breaking approximation for the expected outcome.

scales as $O(\beta)$ as $\beta\to\infty$, while the second has finite limits, the saddle point should be on the curve $0=1+kq^2+(1-k)p^2-2p$. We evaluated numerically m as the function of g. We plot the function $m_{\text{minimax}}(g)$ (solid line on the figure). The exact value of $m_{\text{minimax}}(g)$ is smaller than $m_{\text{spin glass}}(g)$ [where $m_{\text{spin glass}}(g)$ is the maximal value of $\Sigma J_{ij}s_is_j+g\Sigma s_i$], since the minimizer tries to set h_i into directions least favorable for the maximizer, while the constant magnetic field is equivalent to a randomly chosen h_i configuration. However, $m_{\text{minimax}}(g)\geqslant m_{\text{spin glass}}(0)$, since one of $\pm\Sigma h_is_i$ is always nonnegative. $m_{\text{minimax}}(g)\geqslant[g-m_{\text{spin glass}}(0)]$ also holds, since if the spins are set to the same direction as h_i, then the contribution of $\Sigma J_{ij}s_is_j$ cannot be less than $-m_{\text{spin glass}}(0)$ by the symmetry of the couplings J_{ij}. We also expect that $m_{\text{minimax}}(g)$ converges to $g-m_{\text{spin glass}}(0)$ as $g\to\infty$. These considerations provide upper and lower bounds for $m_{\text{minimax}}(g)$ (dotted lines in Fig. 3). Unfortunately, the lower bound is violated for small g, while its asymptotic value is correctly reproduced.

It would be interesting to know if a better, replica symmetry breaking solution would cure this problem. In fact, it is quite possible that even a higher order approximation would not help. The problem is that even at the one stage replica symmetry breaking approximation there is an alternative to Eq. (17):

$$Q_{a\alpha b\alpha}=p,\quad \text{while}\quad Q_{a\alpha b\beta}=q\quad \text{for}\quad \alpha\neq\beta. \tag{19}$$

However, the term $g\Sigma_aH^a\Sigma_\alpha S^{a\alpha}$ is not compatible with the hierarchical structure of $Q_{a\alpha b\beta}$ in Eq. (19), which makes the analytic evaluation of $Z_{n,k}$ very difficult in this case. In principle, the best would be to use the ansatz

$$Q_{a\alpha b\beta}=q_0\quad \text{for}\ a\neq b,\alpha\neq\beta,\quad Q_{a\alpha a\beta}=q_0\quad \text{for}\ \alpha\neq\beta,$$
$$Q_{a\alpha b\alpha}=q_0\quad \text{for}\ a\neq b. \tag{20}$$

The nonultrametric structure of Eq. (29) would make this ansatz very interesting, but it seems hopeless to evaluate $Z_{n,k}$.

IV. MULTIMOVE GAMES

Up to this point only two-move games were treated. The extension for multimove games is straightforward. For example, the outcome of the four-move game

$$M=\inf_u(\sup_v\{\inf_w[\sup_z H(u,v,w,z]\}) \tag{21}$$

is

$$\lim_{\beta_{u,v,w,z}\to\infty}\frac{-1}{\beta_u}\lim_{n\to 0}\left[\left(\sum_{\{u^a,v^{a\alpha},w^{a,\alpha\beta},z^{a\alpha\beta\gamma}\}}\right.\right.$$
$$\left.\left.\times\exp\beta_z\sum_{a\alpha\beta\gamma}H(u^a,v^{a\alpha},w^{a\alpha\beta},z^{a\alpha\beta\gamma})\right)-1\right], \tag{22}$$

where the ranges of the indices are $|a|=n$, $|\alpha|=-\beta_u/\beta_v$, $|\beta|=-\beta_v/\beta_w|\gamma|=-\beta_w/\beta_z$.

The limit $\beta_{u,v,w,z}\to\infty$ corresponds to the optimal strategies of the players. Finite β simulates nonexact optimization, i.e., players with bounded computational capabilities. An interesting case is when one player's temperature is infinite, so the other ones play against random moves. Such games are called ''games against Nature'' [13].

V. DISCUSSION

In the previous sections we used the method of negative replicas to examine optimization problems arising in minimax games. Such games provide examples of very difficult combinatorial problems. In principle our method is able to estimate the expected outcome of some adversary games. Unfortunately, due to the complexity of the calculations emerging in problems of spin-glass type, we manage to treat only fairly simple optimization problems. Nevertheless, the method of negative replicas provides a natural framework to treat game theoretical problems with the machinery of statistical mechanics.

[1] M. Mezard, G. Parisi, and M. A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).

[2] S. Kirkpatrik, C. D. Gelatt, Jr., and M. P. Vecchi, Science **220**, 671 (1983).

[3] M. Mezard and G. Parisi, J. Phys. (France) Lett. **46**, L771 (1985). Reprinted in [1].

[4] Y. T. Fu and P. W. Anderson, J. Phys. A **19**, 1605 (1986). Reprinted in [1].

[5] M. Mezard and G. Parisi, J. Phys. (France) **47**, 1285 (1986). Reprinted in [1].

[6] L. J. Stockmeyer, in *Handbooks in Operational Research and Management Science, Vol. 3: Computing*, edited by E. G.

Coffman, Jr., J. K. Lenstra, A. H. G. Rinnooy Kan (North-Holland, Amsterdam, 1992), pp. 455–517.

[7] V. S. Dotsenko, S. Franz, and M. Mezard, J. Phys. A **27**, 2351 (1994).

[8] V. S. Dotsenko, Phys. Usp. **36**, 455 (1993).

[9] R. W. Penney, A. C. C. Coolen, and D. Sherrington, J. Phys. A **26**, 3681 (1993).

[10] A. C. C. Coolen, D. Sherrington, and R. W. Penney, Phys. Rev. B **48**, 16 116 (1993).

[11] D. E. Feldman and V. S. Dotsenko, J. Phys. A **27**, L821 (1994).

[12] G. Parisi, Phys. Lett. **73A**, 203 (1979).

[13] C. H. Papadimitriou J. Comput. Syst. Sci. **31**, 288 (1985).